Convex Optimization

(P) $\min_{x \in \mathbb{R}^n} f(x)$ subject to $\begin{cases} f_i(x) \leq 0 & i = 1 \dots p \\ f_j(x) = 0 & j = p+1, \dots, m \\ x \in C \end{cases}$

$$S := \{ x \in C \mid f_i(x) \leq 0,\ f_j(x) = 0,\ i = 1 \dots p,\ j = p+1 \dots m \}$$

Goal: Derive condition which allow to decide whether a point $x \in S$ is optimal or not

$\longrightarrow$ basics for many numerical recepies.

Assumptions: $\forall\, i = 1 \dots m$:

(A1) $C \subseteq \mathbb{R}^n$ convex and $C \subseteq \operatorname{supp} f_i$

(A2) $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex for $i = 1 \dots p$

(A3) $f_j : \mathbb{R}^n \to \mathbb{R}$ affine for $j = p+1 \dots m$

(Aconvex) $:=$ (A1) $\wedge$ (A2) $\wedge$ (A3).

REM: (A1)–(A3) $\Rightarrow$ S convex and furthermore if C closed $f_i|_C$, $f_j|_C$ continuous (implied by convexity) $\Rightarrow$ S closed

(A4) a) $\exists\, \bar{x} \in S \cap \mathring{C}$

b) $\forall\, f_i$, $i = 1 \dots p$, not affine $\exists\, x_i \in S$: $f(x_i) < 0$

Assumption (A4) is called Slater cond.

(Aslater) $:=$ (A4) a) $\wedge$ b)

The strategy will be based on the hyperplane seperation LEM:

LEM: (Hyperplane separation)

$A \neq \emptyset$, $0 \notin A$, A convex, $A \subseteq \mathbb{R}^{m+1}$

$\Rightarrow \exists\, z \in \mathbb{R}^{m+1}$, $z \neq 0$ s.t.

i) $\forall v \in A: \; z \cdot v \geq 0$

ii) $\exists\, \bar{v} \in A: \; z \cdot \bar{v} > 0$

This LEM allows to show for our optimization program:

LEM:

(i) Let (P) fulfill (Aconvex) and let $\alpha := \inf_{x \in S} f_0(x) \in \mathbb{R}$

Only functions P assumed! Not $\exists\, x^* \in C$ with $f_0(x^*) = \alpha$!

then:

$\exists\, z \in \mathbb{R}^{m+1}$, $z \neq 0$, $z_i \geq 0$ $i = 1 \ldots p$

such that

$\forall x \in C: \; z_0 \left[f_0(x) - \alpha \right] + \sum_{i=1}^{m} z_i f_i(x) \geq 0$

(ii) If in addition (Aslater)

$\exists\, y \in \mathbb{R}^m$, $y_i \geq 0$ $i = 1 \ldots p$

such that

$\forall x \in C: \; \left(f_0(x) - \alpha \right) + \sum_{i=1}^{m} y_i f_i(x) \geq 0$

Implications

Let x^* be an optimal solution of (P)

$\Rightarrow \alpha = f(x^*)$ and $\alpha = \inf_{x \in S} f_0(x)$

(Aconvex) $\overset{\text{LEM}(i)}{\Rightarrow} \exists\, z \in \mathbb{R}^{m+1}$, $z \neq 0$, $z_i \geq 0$ for $i = 1 \ldots m$ s.t.

$$z_0 \left[f_0(x) - f_0(x^*) \right] + \sum_{i=1}^{m} z_i f_i(x) \geq 0 \quad (*)$$

Suppose f_i for $i = 1 \ldots m$ are diff., then

(I) $\sum_{i=0}^{m} z_i \nabla f_i(x^*) = 0$

(II) $f_i(x^*) z_i = 0$, $f_i(x^*) \leq 0$, $z_i \geq 0$ for $i = 1 \ldots p$

(III) $f_j(x^*) = 0$ for $j = p+1 \ldots m$

Because: (I)

- $\phi(x) = z_0 (f_0(x) - f_0(x^*)) + \sum_{i=1}^{m} z_i f_i(x)$

  is convex $\wedge$ diff. $\wedge$ $\phi(x^*) \leq 0$

- but $(*) \Rightarrow \phi(x) \geq 0$

  $\Rightarrow x^*$ gives rise to loc. min. of $\phi(x)$

  $\Rightarrow \nabla_x \phi(x)\big|_{x=x^*} = 0$

  by Fermat's theorem

(I) $(*) \Rightarrow z_i \geq 0 \quad \forall\, i = 1 \ldots p$

but $f_i(x^*) \leq 0$ for $i = 1 \ldots p$

Assume $z_k f_k(x^*) \neq 0$

$\Rightarrow z_k f_k(x^*) < 0$

$\Rightarrow \phi(x^*) < 0$ ↯ $\Rightarrow z_k f_k(x^*) = 0$.

(II) directly

And in case (Aslater) holds, too,

we may choose $y = \frac{z}{z_0}$ because $z_0 \neq 0$:

Thm: Let (P) fulfill (Aconvex) $\wedge$ (Aslater).

If $x^* \in S$ is optimal solution,

f_i are diff at x^* $i = 1 \ldots m$

then $\exists\, y \in \mathbb{R}^m$, $y_i \geq 0$ s.t.

$$(KKT) \begin{cases} (I) & \sum_{i=1}^{m} y_i \nabla f_i(x^*) + \nabla f_0(x^*) = 0 \\ (II) & f_i(x^*)\, y_i = 0,\ f_i(x^*) \leq 0,\ y_i \geq 0 \\ & \text{for } i = 1 \ldots p \\ (III) & f_j(x^*) = 0 \text{ for } j = p+1, \ldots, m. \end{cases}$$